On the eigenvalue problem for arbitrary odd elements of the Lie superalgebra $\mathfrak{gl}(1|n)$ and applications

On the eigenvalue problem for arbitrary odd elements of the Lie superalgebra $\mathfrak{gl}(1|n)$ and applications

S Lievens, N I Stoilova[1] **and J Van der Jeugt**

Department of Applied Mathematics and Computer Science, Ghent University, Krijgslaan 281-S9, B-9000 Gent, Belgium

E-mail: Stijn.Lievens@UGent.be, Neli.Stoilova@UGent.be and Joris.VanderJeugt@UGent.be

Abstract

In a Wigner quantum mechanical model, with a solution in terms of the Lie superalgebra $\mathfrak{gl}(1|n)$, one is faced with determining the eigenvalues and eigenvectors for an arbitrary self-adjoint odd element of $\mathfrak{gl}(1|n)$ in any unitary irreducible representation W. We show that the eigenvalue problem can be solved by the decomposition of W with respect to the branching $\mathfrak{gl}(1|n) \to \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(n-1)$. The eigenvector problem is much harder, since the Gel'fand–Zetlin basis of W is involved, and the explicit actions of $\mathfrak{gl}(1|n)$ generators on this basis are fairly complicated. Using properties of the Gel'fand–Zetlin basis, we manage to present a solution for this problem as well. Our solution is illustrated for two special classes of unitary $\mathfrak{gl}(1|n)$ representations: the so-called Fock representations and the ladder representations.

PACS numbers: 03.65.−w, 03.65.Fd, 02.20.−a

1. Introduction

Recently, the Wigner quantum approach of a quantum mechanical model consisting of a linear chain of n identical harmonic oscillators coupled by some nearest neighbour interaction was considered [1]. In the standard approach, where the canonical commutation relations between position and momentum operators are required, a solution of the system is well known [2]. In [1], it was shown that these requirements can be relaxed and the problem was treated as a Wigner quantum system. As a consequence, the system allows besides the canonical solution also other types of solutions. In particular, it was shown that the (finite-dimensional) unitary irreducible representations of the Lie superalgebra $\mathfrak{gl}(1|n)$ give rise to new solutions.

---

[1] Permanent address: Institute for Nuclear Research and Nuclear Energy Boul. Tsarigradsko Chaussee 72, 1784 Sofia, Bulgaria.

In order to study properties of these new solutions, one is faced with some computationally difficult problems in the representation theory of $\mathfrak{gl}(1|n)$ [3, 4]. More precisely, consider the standard basis of $\mathfrak{gl}(1|n)$ consisting of elements e_{ij} ($0 \leqslant i, j \leqslant n$), with e_{0j} and e_{j0} ($1 \leqslant j \leqslant n$) the odd elements of the Lie superalgebra, with bracket (2.8), and with star condition $e_{ij}^{\dagger} = e_{ji}$. The unitary representations $W = W([m]_{n+1})$ of $\mathfrak{gl}(1|n)$ are well known [5]: they are labelled by some $(n+1)$-tuple $[m]_{n+1}$ subject to certain conditions. Even more: for such representations, a Gel'fand–Zetlin basis has been constructed and the explicit action of the $\mathfrak{gl}(1|n)$ generators on the basis vectors of W is also known [6]. Explicit actions of generators on a Gel'fand–Zetlin basis (GZ basis) are usually quite involved, and this is also the case for $\mathfrak{gl}(1|n)$. In particular, the action of the odd generators e_{0j} and e_{j0} on a GZ-basis vector is very complicated, see (3.9)–(3.8).

The operators we intend to study are the position and momentum operators $\hat{q}_r$ and $\hat{p}_r$ ($r = 1, \ldots, n$) of the quantum system. These are self-adjoint operators, and in the $\mathfrak{gl}(1|n)$ solution of the problem considered in [1] as a Wigner quantum system their expression is of the form

$$\sum_{j=1}^{n} \alpha_j e_{0j} + \sum_{j=1}^{n} \alpha_j^* e_{j0}, \tag{1.1}$$

for certain constants α_j. This is an arbitrary self-adjoint odd element in $\mathfrak{gl}(1|n)$. For such elements, we want to determine the spectrum (eigenvalues) in any unitary representation W. Furthermore, we wish to construct an explicit set of orthonormal eigenvectors of (1.1) in terms of the GZ basis of W. The eigenvalue problem turns out to be feasible, thanks to group theoretical methods. In fact, we show how it is related to the decomposition of $\mathfrak{gl}(1|n)$ representations into representations of the subalgebra $\mathfrak{gl}(1|1) \oplus \mathfrak{gl}(n-1)$. The eigenvector problem is much harder, as one is faced with the sophisticated action of the Lie superalgebra generators on the GZ-basis vectors. But also here, we manage to present a solution.

The structure of the paper is as follows. In section 2, we describe in more detail the origin of the problem. We also recall the structure of the GZ basis for $\mathfrak{gl}(1|n)$ representations, and the conditions for unitarity. In section 3, we convert the general eigenvalue problem to a simpler problem by switching to another set of odd generators for the Lie superalgebra $\mathfrak{gl}(1|n)$. In terms of the new set of generators, (1.1) has a simple expression: in fact it becomes an element of a $\mathfrak{gl}(1|1)$ subalgebra of $\mathfrak{gl}(1|n)$. The branching $\mathfrak{gl}(1|n) \rightarrow \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(n-1)$, studied in section 4, leads to an answer of the eigenvalue problem. In the following section, we construct the essential relation that expresses the highest weight vector of W with respect to the new set of generators in terms of the 'old' GZ-basis vectors. Combining this with the known actions on such GZ-basis vectors yields a solution for the eigenvector problem. Then we illustrate our results for two important classes of unitary representations. Section 6 deals with Fock representations of $\mathfrak{gl}(1|n)$. These representations are quite simple, and have been considered in [7]. Nevertheless, the eigenvector problem turned out to be difficult and was left as an open problem in [1]. With the techniques developed in this paper, a simple solution to the eigenvector problem is obtained. Section 7 deals with another relatively simple class of representations, the ladder representations of $\mathfrak{gl}(1|n)$. Also here, we illustrate how our techniques lead to a complete solution of the eigenvalue and eigenvector problem. We conclude the paper by some final remarks.

2. Description of the problem

In several models [8–12] a quantum system consisting of a linear chain of n identical harmonic oscillators coupled by springs is used. The Hamiltonian of such a system is given by

$$\hat{H} = \sum_{r=1}^{n} \left(\frac{\hat{p}_r^2}{2\mu} + \frac{\mu\omega^2}{2}\hat{q}_r^2 + \frac{c\mu}{2}(\hat{q}_r - \hat{q}_{r+1})^2 \right), \tag{2.1}$$

where each oscillator has mass μ and frequency ω, $\hat{q}_r$ and $\hat{p}_r$ stand for the position and momentum operator for the rth oscillator (or rather, $\hat{q}_r$ measures the displacement of the rth mass point with respect to its equilibrium position), and $c > 0$ is the coupling strength. Often, one assumes periodic boundary conditions (also in this paper), i.e.

$$\hat{q}_{n+1} \equiv \hat{q}_1. \tag{2.2}$$

In the solution for such a system, one introduces finite Fourier transforms of the (self-adjoint) operators $\hat{q}_r$ and $\hat{p}_r$ by

$$\hat{q}_r = \sum_{j=1}^{n} \sqrt{\frac{\hbar}{2\mu n \omega_j}} \left(e^{-2\pi \mathrm{i} jr/n} a_j^+ + e^{2\pi \mathrm{i} jr/n} a_j^- \right), \tag{2.3}$$

$$\hat{p}_r = \sum_{j=1}^{n} \mathrm{i}\sqrt{\frac{\mu \omega_j \hbar}{2n}} \left(e^{-2\pi \mathrm{i} jr/n} a_j^+ - e^{2\pi \mathrm{i} jr/n} a_j^- \right), \tag{2.4}$$

where ω_j are positive numbers with

$$\omega_j^2 = \omega^2 + 2c - 2c\cos\left(\frac{2\pi \mathrm{j}}{n}\right) = \omega^2 + 4c\sin^2\left(\frac{\pi \mathrm{j}}{n}\right), \tag{2.5}$$

and $a_j^{\pm}$ are operators satisfying $\left(a_j^{\pm}\right)^{\dagger} = a_j^{\mp}$. In terms of these new operators, the Hamiltonian reads

$$\hat{H} = \sum_{j=1}^{n} \frac{\hbar\omega_j}{2}(a_j^- a_j^+ + a_j^+ a_j^-). \tag{2.6}$$

If one assumes the canonical commutation relations for the operators $\hat{q}_r$ and $\hat{p}_r$, then the operators $a_j^{\pm}$ satisfy the usual boson relations $\left[a_j^{\pm}, a_k^{\pm}\right] = 0$, $\left[a_j^-, a_k^+\right] = \delta_{jk}$, and the corresponding solutions are easy to describe. In [1], however, it was shown that one can relax the canonical commutation relations for this system, and instead approach it as a Wigner quantum system [13]–[15], leading to other classes of solutions besides the canonical ones. In this approach, the canonical commutation relations are not required but replaced by the quantization relations following from the compatibility between Hamilton's equations and the Heisenberg equations. Explicitly, these relations are [1]

$$\left[\sum_{j=1}^{n} \omega_j \left(a_j^- a_j^+ + a_j^+ a_j^-\right), a_k^{\pm} \right] = \pm 2\omega_k a_k^{\pm}, \qquad (k = 1, 2\ldots, n). \tag{2.7}$$

These are triple relations involving anticommutators and commutators, and it was shown [1] that such relations have a solution in terms of generators of the Lie superalgebra $\mathfrak{gl}(1|n)$ [3, 4]. More explicitly, let $\mathfrak{gl}(1|n)$ be the Lie superalgebra with standard basis elements e_{jk} $(j, k = 0, 1, \ldots, n)$, where e_{k0} and e_{0k} $(k = 1, \ldots, n)$ are odd elements and the remaining basis elements are even, with bracket

$$[\![e_{ij}, e_{kl}]\!] = \delta_{jk}e_{il} - (-1)^{\deg(e_{ij})\deg(e_{kl})}\delta_{il}e_{kj}, \tag{2.8}$$

and star condition $e_{ij}^{\dagger} = e_{ji}$. Then a solution of (2.7) is provided by

$$a_j^- = \sqrt{\frac{2\beta_j}{\omega_j}}e_{j0}, \qquad a_j^+ = \sqrt{\frac{2\beta_j}{\omega_j}}e_{0j}, \qquad (j = 1, \ldots, n), \tag{2.9}$$

where

$$\beta_j = -\omega_j + \frac{1}{n-1}\sum_{k=1}^{n}\omega_k, \qquad (j = 1, \ldots, n). \tag{2.10}$$

All these numbers β_j should be non-negative, and in [1] we have shown that this is possible provided the coupling constant c is bounded by some critical value c_0 (which we shall assume to be the case here). So, for this $\mathfrak{gl}(1|n)$ solution, one finds

$$\hat{q}_r = \sqrt{\frac{\hbar}{\mu n}}\sum_{j=1}^{n}(\gamma_j\,\mathrm{e}^{-2\pi\mathrm{i}jr/n}e_{0j} + \gamma_j\,\mathrm{e}^{2\pi\mathrm{i}jr/n}e_{j0}), \tag{2.11}$$

$$\hat{p}_r = \mathrm{i}\sqrt{\frac{\mu\hbar}{n}}\sum_{j=1}^{n}(\sqrt{\beta_j}\,\mathrm{e}^{-2\pi\mathrm{i}jr/n}e_{0j} - \sqrt{\beta_j}\,\mathrm{e}^{2\pi\mathrm{i}jr/n}e_{j0}), \tag{2.12}$$

where we introduce yet another set of positive numbers

$$\gamma_j = \sqrt{\beta_j}/\omega_j \qquad (j = 1, \ldots, n) \qquad \text{and} \quad \gamma = \gamma_1^2 + \cdots + \gamma_n^2. \tag{2.13}$$

Equations (2.11) and (2.12) give a description of the 'physical operators' $\hat{q}_r$ and $\hat{p}_r$ in terms of $\mathfrak{gl}(1|n)$ generators. In order to study properties of such operators (spectra or eigenvalues, eigenvectors), one should consider representations of $\mathfrak{gl}(1|n)$ for which the star condition $e_{ij}^{\dagger} = e_{ji}$ is satisfied. These are the star representations or unitary representations $W([m]_{n+1})$ of $\mathfrak{gl}(1|n)$, and they are well known [5].

   As (2.11) and (2.12) are the 'physical operators' corresponding to position and momentum of the rth oscillator, we are interested in the following problems:

(a) describe the eigenvalues of $\hat{q}_r$ and $\hat{p}_r$ in any unitary representation $W([m]_{n+1})$;
(b) construct the eigenvectors of $\hat{q}_r$ and $\hat{p}_r$ in $W([m]_{n+1})$.

At first sight, these problems might look easy. However, a closer look at the explicit actions of the $\mathfrak{gl}(1|n)$ generators e_{0j} and e_{j0} on the GZ basis, see equations (2.25) and (2.26) in [6] or (3.9)–(3.8) in the appendix, shows that these expressions are extremely complicated. Since (2.11) and (2.12) are linear combinations of the generators e_{0j} and e_{j0}, one could expect that the answer to the above questions gives rise to unfeasible computations.

   Nonetheless, we shall show that a group theoretical approach (using subalgebras, branching rules, and a proper use of two inequivalent GZ-bases) leads to a solution for these two problems.

   We end this section by describing the relevant representations, i.e. the unitary irreducible representations $W([m]_{n+1})$ of $\mathfrak{gl}(1|n)$ [5], and their GZ basis [6]; see also [16, 17]. The finite-dimensional irreducible representations (simple modules) $W([m]_{n+1})$ of the Lie superalgebra $\mathfrak{gl}(1|n)$ are in one-to-one correspondence with the set of all complex $(n+1)$-tuples [3, 4]

$$[m]_{n+1} = [m_{0,n+1}, m_{1,n+1}, \ldots, m_{n,n+1}], \tag{2.14}$$

for which

$$m_{i,n+1} - m_{j,n+1} \in \mathbb{Z}_+ \qquad (1 \leqslant i < j \leqslant n). \tag{2.15}$$

In a standard weight space basis, the highest weight Λ of $W([m]_{n+1})$ is given by

$$\Lambda = m_{0,n+1}\epsilon + \sum_{i=1}^{n}m_{i,n+1}\delta_i. \tag{2.16}$$

A Gel'fand–Zetlin basis for the $\mathfrak{gl}(1|n)$ representation $W([m]_{n+1})$ has been given and discussed in [6, proposition 2], where it was shown that the set of vectors

$$
|m)_e =
\begin{vmatrix}
m_{0,n+1} & m_{1,n+1} & \cdots & m_{n-2,n+1} & m_{n-1,n+1} & m_{n,n+1} \\
 & m_{1n} & \cdots & m_{n-2,n} & m_{n-1,n} & m_{nn} \\
 & m_{1,n-1} & \cdots & m_{n-2,n-1} & m_{n-1,n-1} & \\
 & \vdots & \iddots & & & \\
 & m_{11} & & & &
\end{vmatrix}_e
\tag{2.17}
$$

satisfying the conditions

(GZ1) $m_{i,n+1}$ are fixed and $m_{i,n+1} - m_{j,n+1} \in \mathbb{Z}_+$ $(1 \leqslant i < j \leqslant n)$,
(GZ2) $m_{in} - m_{i,n+1} = \theta_i \in \{0, 1\}$ $(1 \leqslant i \leqslant n)$,
(GZ3) if for $k \in \{1, \ldots, n\}$ one has $m_{0,n+1} + m_{k,n+1} = k - 1$ then $\theta_k = 0$,
(GZ4) $m_{i,j+1} - m_{ij} \in \mathbb{Z}_+$ and $m_{ij} - m_{i+1,j+1} \in \mathbb{Z}_+$ $(1 \leqslant i \leqslant j \leqslant n - 1)$,

constitute a basis in $W([m]_{n+1})$. We have added here a subscript e to the vectors $|m)_e$ in order to distinguish them from another basis $|m)_E$ for $W([m]_{n+1})$ which will be introduced later. For the explicit action of a set of $\mathfrak{gl}(1|n)$ generators on the basis vectors (2.17), see (A.1)–(A.6). Following (2.14), it will be convenient to denote the elements of the other rows in $|m)_e$, or more generally k-tuples, by

$$
[m]_k = [m_{1k}, m_{2k}, \ldots, m_{kk}], \qquad (k = 1, \ldots, n).
\tag{2.18}
$$

With respect to the inner product $\langle |m')_e, |m)_e \rangle = \delta_{m,m'}$ and the condition $e_{ij}^\dagger = e_{ji}$, the representations $W([m]_{n+1})$ are unitary if and only if one of the following conditions is satisfied [6, proposition 3].

(U1) The highest weight is real and

$$
m_{0,n+1} + m_{n,n+1} - n + 1 > 0.
\tag{2.19}
$$

In this case, the representation is typical.
(U2) The highest weight is real and there exists a $k \in \{1, 2, \ldots, n\}$ such that

$$
m_{0,n+1} + m_{k,n+1} = k - 1, \qquad m_{k,n+1} = m_{k+1,n+1} = \cdots = m_{n,n+1}.
\tag{2.20}
$$

In this case, the representation is atypical of type k.

Note that the highest weight vector of $W([m]_{n+1})$, denoted by $|\Lambda)_e$, is given by

$$
|\Lambda)_e =
\begin{vmatrix}
m_{0,n+1} & m_{1,n+1} & \cdots & m_{n-2,n+1} & m_{n-1,n+1} & m_{n,n+1} \\
 & m_{1,n+1} & \cdots & m_{n-2,n+1} & m_{n-1,n+1} & m_{n,n+1} \\
 & m_{1,n+1} & \cdots & m_{n-2,n+1} & m_{n-1,n+1} & \\
 & \vdots & \iddots & & & \\
 & m_{1,n+1} & & & &
\end{vmatrix}_e .
\tag{2.21}
$$

Finally, note that the GZ-basis vectors $|m)_e$ are stationary states of the quantum system. Indeed, by (2.6), (2.8) and (2.9), one has

$$
\hat{H} = \hbar \left(\left(\sum_{j=1}^n \beta_j \right) e_{00} + \sum_{j=1}^n \beta_j e_{jj} \right),
\tag{2.22}
$$

so one finds, using (A.1)–(A.2):

$$
\hat{H} |m)_e = \hbar \tilde{E}_m |m)_e,
\tag{2.23}
$$

with

$$
\tilde{E}_m = \left(\sum_{j=1}^n \beta_j \right) \left(\sum_{l=0}^n m_{l,n+1} - \sum_{l=1}^n m_{ln} \right) + \sum_{j=1}^n \beta_j \left(\sum_{l=1}^j m_{lj} - \sum_{l=1}^{j-1} m_{l,j-1} \right).
\tag{2.24}
$$

3. Another set of $\mathfrak{gl}(1|n)$ generators

The purpose is to describe the eigenvalues of $\hat{q}_r$ and $\hat{p}_r$, and to give their eigenvectors in terms of the GZ-basis vectors $|m\rangle_e$ of $W([m]_{n+1})$, i.e. in terms of the stationary states. The structure of $\hat{q}_r$ and $\hat{p}_r$ in terms of the generators e_{0j} and e_{j0} is similar, see (2.11) and (2.12), so it is sufficient to concentrate on $\hat{q}_r$ only, with

$$\hat{q}_r = \sqrt{\frac{\hbar}{\mu n}} \sum_{j=1}^{n} (\gamma_j \, e^{-2\pi \mathrm{i}jr/n} e_{0j} + \gamma_j \, e^{2\pi \mathrm{i}jr/n} e_{j0}). \tag{3.1}$$

Since the description of $\hat{q}_r$ in terms of e_{0j} and e_{j0} is complicated (at least for the action on GZ-basis vectors), we shall switch to another set of $\mathfrak{gl}(1|n)$ generators. For this purpose, recall the following proposition [18].

Proposition 1. *The Lie superalgebra generated by $2n$ odd elements $\tilde{e}_{j0}$ and $\tilde{e}_{0j}$, with $1 \leqslant j \leqslant n$, subject to the relations*

$$\{\tilde{e}_{j0}, \tilde{e}_{k0}\} = \{\tilde{e}_{0j}, \tilde{e}_{0k}\} = 0, \tag{3.2}$$

$$[\{\tilde{e}_{j0}, \tilde{e}_{0k}\}, \tilde{e}_{l0}] = \delta_{kl}\tilde{e}_{j0} - \delta_{jk}\tilde{e}_{l0}, \tag{3.3}$$

$$[\{\tilde{e}_{j0}, \tilde{e}_{0k}\}, \tilde{e}_{0l}] = \delta_{jk}\tilde{e}_{0l} - \delta_{lj}\tilde{e}_{0k}, \tag{3.4}$$

is isomorphic to $\mathfrak{sl}(1|n)$.

So clearly, our standard elements e_{j0} and e_{0j} generate $\mathfrak{sl}(1|n)$. The only difference between $\mathfrak{sl}(1|n)$ and $\mathfrak{gl}(1|n)$ comes from the Cartan subalgebra: for $\mathfrak{gl}(1|n)$ this is spanned by all elements e_{jj} ($0 \leqslant j \leqslant n$), and for $\mathfrak{sl}(1|n)$ by $e_{00} + e_{jj}$ ($1 \leqslant j \leqslant n$).

The following proposition is easy but essential in our analysis.

Proposition 2. *Let $U = (U_{jl})_{1 \leqslant j \leqslant n, 1 \leqslant l \leqslant n}$ be a unitary $n \times n$ matrix, and let*

$$E_{j0} = \sum_{l=1}^{n} U_{jl}e_{l0} \qquad and \quad E_{0j} = \sum_{l=1}^{n} U_{jl}^{*}e_{0l} \qquad (1 \leqslant j \leqslant n). \tag{3.5}$$

Then the elements E_{j0} and E_{0j} satisfy the same defining relations (3.2)–(3.4) as the elements e_{j0} and e_{0j}. In other words, also the E_{j0} and E_{0j} generate $\mathfrak{sl}(1|n)$.

Proof. It is a simple exercise to verify that the elements E_{j0} and E_{0j} satisfy the relations (3.2)–(3.4). As an example, consider

$$
\begin{aligned}
[\{E_{j0}, E_{0k}\}, E_{l0}] &= \sum_{i_1,i_2,i_3} U_{ji_1} U_{ki_2}^{*} U_{li_3} \big[\{e_{i_10}, e_{0i_2}\}, e_{i_30} \big] \\
&= \sum_{i_1,i_2,i_3} U_{ji_1} U_{ki_2}^{*} U_{li_3} \big(\delta_{i_2i_3} e_{i_10} - \delta_{i_1i_2} e_{i_30} \big) \\
&= \sum_{i_2} U_{ki_2}^{*} U_{li_2} \sum_{i_1} U_{ji_1} e_{i_10} - \sum_{i_2} U_{ji_2} U_{ki_2}^{*} \sum_{i_3} U_{li_3} e_{i_30} \\
&= \sum_{i_2} U_{li_2} U_{i_2k}^{\dagger} \sum_{i_1} U_{ji_1} e_{i_10} - \sum_{i_2} U_{ji_2} U_{i_2k}^{\dagger} \sum_{i_3} U_{li_3} e_{i_30} \\
&= \delta_{lk} E_{j0} - \delta_{jk} E_{l0}.
\end{aligned}
$$

$\square$

Using this proposition, it will be useful to identify the two parts of (3.1), $\sum_{j=1}^n \gamma_j \, \mathrm{e}^{-2\pi \mathrm{i} jr/n} e_{0j}$ and $\sum_{j=1}^n \gamma_j \, \mathrm{e}^{2\pi \mathrm{i} jr/n} e_{j0}$ as single generators E_{0k} and E_{k0} for some k. Hence, let us define

$$E_{n0} = \frac{1}{\sqrt{\gamma_1^2 + \cdots + \gamma_n^2}} \sum_{j=1}^n \gamma_j \, \mathrm{e}^{2\pi \mathrm{i} jr/n} e_{j0} = \frac{1}{\sqrt{\gamma}} \sum_{j=1}^n \gamma_j \, \mathrm{e}^{2\pi \mathrm{i} jr/n} e_{j0}, \qquad (3.6)$$

$$E_{0n} = \frac{1}{\sqrt{\gamma_1^2 + \cdots + \gamma_n^2}} \sum_{j=1}^n \gamma_j \, \mathrm{e}^{-2\pi \mathrm{i} jr/n} e_{0j} = \frac{1}{\sqrt{\gamma}} \sum_{j=1}^n \gamma_j \, \mathrm{e}^{-2\pi \mathrm{i} jr/n} e_{0j}. \qquad (3.7)$$

Note that we have divided by $\sqrt{\gamma}$, so that the coefficients are entries of a unitary matrix U, as required in (3.5). Next, we should supplement (3.6) and (3.7) by other linear combinations of the e_{j0} and e_{0j}, such that the transition matrix is unitary. In principle, any matrix U with last row $U_{nj} = \gamma_j \, \mathrm{e}^{2\pi \mathrm{i} jr/n}/\sqrt{\gamma}$ could be proposed. However, in order to make computations for eigenvectors easier, we will propose a matrix U that is as simple as possible, i.e. with as many zero entries as possible. Note that one cannot make a triangular choice for U, since the only triangular matrix that is also unitary is diagonal. So we will make a choice that is as close as possible to a triangular matrix, namely a Hessenberg matrix, so that all entries U_{jl} with $l > j + 1$ are zero. This leads to the following expressions, for $j = 1, 2, \ldots, n - 1$:

$$E_{j0} = \frac{1}{\sqrt{\frac{1}{\gamma_1^2 + \cdots + \gamma_j^2} + \frac{1}{\gamma_{j+1}^2}}} \left(\sum_{l=1}^j \frac{\mathrm{e}^{2\pi \mathrm{i} rl/n}}{\gamma_1^2 + \cdots + \gamma_j^2} \gamma_l e_{l0} - \frac{1}{\gamma_{j+1}} \mathrm{e}^{2\pi \mathrm{i} r(j+1)/n} e_{j+1,0} \right), \qquad (3.8)$$

$$E_{0j} = \frac{1}{\sqrt{\frac{1}{\gamma_1^2 + \cdots + \gamma_j^2} + \frac{1}{\gamma_{j+1}^2}}} \left(\sum_{l=1}^j \frac{\mathrm{e}^{-2\pi \mathrm{i} rl/n}}{\gamma_1^2 + \cdots + \gamma_j^2} \gamma_l e_{0l} - \frac{1}{\gamma_{j+1}} \mathrm{e}^{-2\pi \mathrm{i} r(j+1)/n} e_{0, j+1} \right). \qquad (3.9)$$

It is a simple exercise to verify that the transition matrix U defined by means of (3.8) and (3.6) is indeed a unitary matrix. So operators (3.6)–(3.9) form a set of generators for $\mathfrak{sl}(1|n)$, such that the position operator $\hat{q}_r$ becomes

$$\hat{q}_r = \sqrt{\frac{\hbar \gamma}{\mu n}} (E_{0n} + E_{n0}). \qquad (3.10)$$

Note that for every different position operator $\hat{q}_r$ (i.e. for every different r), one has a *different set of generators*, so we should denote them by $E_{j0}^{(r)}$ and $E_{0j}^{(r)}$. This overloads the notation, however. So we shall assume that r is fixed, and drop the superscript (r) from the generators.

The elements E_{0j} and E_{j0} generate $\mathfrak{sl}(1|n)$. The new odd basis elements of $\mathfrak{sl}(1|n)$ are directly given by (3.6)–(3.9). The new even basis elements of $\mathfrak{sl}(1|n)$ are of the form $E_{jk} = \{E_{j0}, E_{0k}\}$ with $j \neq k$, and $\{E_{j0}, E_{0j}\}$. Since, without writing the matrix elements of U explicitly as in (3.6)–(3.9), for $j = 1, 2, \ldots, n$:

$$\{E_{j0}, E_{0j}\} = e_{00} + \sum_{l=1}^n \sum_{k=1}^n U_{jl} U_{jk}^* e_{lk} \qquad (3.11)$$

one can extend the new $\mathfrak{sl}(1|n)$ basis to a $\mathfrak{gl}(1|n)$ basis by putting $E_{00} = e_{00}$, and E_{jj} equal to the remaining part in (3.11), i.e. $E_{jj} = \{E_{j0}, E_{0j}\} - e_{00}$.

So we have a *new basis* E_{ij} for $\mathfrak{gl}(1|n)$, satisfying the same relations (2.8) as the old basis e_{ij}, and the same star conditions $E_{ij}^\dagger = E_{ji}$. In terms of this new basis, the position

operator $\hat{q}_r$ has a simple expression, see (3.10). Due to this simple expression, the eigenvalues and eigenvectors of $\hat{q}_r$ can be computed. With respect to this new basis, the representation $W([m]_{n+1})$ has a new highest weight vector, to be denoted by $|\Lambda\rangle_E$. One essential task will be the expansion of $|\Lambda\rangle_E$ in terms of the old GZ-basis vectors $|m\rangle_e$. Also with respect to this new basis E_{ij}, one can define a new GZ basis for $W([m]_{n+1})$, the vectors of this basis being denoted by $|m\rangle_E$. The action of E_{ij} on vectors $|m\rangle_E$ is identical to the action of e_{ij} on vectors $|m\rangle_e$.

In the following section, we shall consider the branching $\mathfrak{gl}(1|n) \to \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(n-1)$ for $W([m]_{n+1})$, with respect to this new basis. This will yield the eigenvalues of $\hat{q}_r$ in $W([m]_{n+1})$. The (orthonormal) eigenvectors of $\hat{q}_r$ are easy to describe in the $|m\rangle_E$ basis of $W([m]_{n+1})$. In section 5, we make the connection between the old basis vectors $|m\rangle_e$ and the new ones $|m\rangle_E$, leading to a description of the $\hat{q}_r$ eigenvectors in the original basis.

4. On the decomposition $\mathfrak{gl}(1|n) \to \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(n-1)$ for unitary representations

Consider the Lie superalgebra $\mathfrak{gl}(1|n)$ with (new) basis elements E_{ij} $(i, j = 0, 1, \ldots, n)$ satisfying the standard relations (2.8). We consider the finite-dimensional unitary irreducible representations $W([m]_{n+1})$ with GZ-basis vectors $|m\rangle_E$. The action of E_{ij} on $|m\rangle_E$ is identical to that of e_{ij} on $|m\rangle_e$ (see (A.1)–(A.6)). In particular, the diagonal action reads

$$E_{00}|m\rangle_E = \left(m_{0,n+1} - \sum_{j=1}^{n} \theta_j \right) |m\rangle_E;$$

$$E_{jj}|m\rangle_E = \left(\sum_{l=1}^{j} m_{lj} - \sum_{l=1}^{j-1} m_{l,j-1} \right) |m\rangle_E, \qquad (1 \leqslant j \leqslant n). \tag{4.1}$$

In order to describe the decomposition $\mathfrak{gl}(1|n) \to \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(n-1)$ for such unitary representations, let us first list (and fix the notation for) the unitary representations of $\mathfrak{gl}(1|1)$ [6]. Let $\{e_{00}, e_{10}, e_{01}, e_{11}\}$ be a basis for $\mathfrak{gl}(1|1)$ and denote the highest weight labels $[m_{0,2}, m_{1,2}]$ by $[a, b]$ and the representation itself by $W([a, b])$. Then, following (U1)–(U2), there are two possibilities.

(1) A typical unitary $\mathfrak{gl}(1|1)$ representation $W([a, b])$, with $a, b \in \mathbb{R}$ and $a + b > 0$. The GZ basis of the representation consists of two vectors only, which we shall denote by v and w, and the action is given by

$$\begin{aligned}
e_{00}v &= av, & e_{00}w &= (a-1)w, \\
e_{11}v &= bv, & e_{11}w &= (b+1)w, \\
e_{01}v &= 0, & e_{01}w &= \sqrt{a+b}\,v, \\
e_{10}v &= \sqrt{a+b}\,w, & e_{10}w &= 0.
\end{aligned} \tag{4.2}$$

The weights of the representation are (a, b) and $(a-1, b+1)$.

(2) An atypical unitary $\mathfrak{gl}(1|1)$ representation $W([a, b])$, with $a, b \in \mathbb{R}$ and $a + b = 0$. The GZ basis consists of one vector only, denoted by v, and the only non-zero actions are

$$e_{00}v = av, \qquad e_{11}v = -av. \tag{4.3}$$

The weight of the representation is $(a, -a)$.

The new GZ basis (and the new basis E_{ij} for $\mathfrak{gl}(1|n)$) can now be used to find the decomposition $\mathfrak{gl}(1|n) \to \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(n-1)$ for unitary $\mathfrak{gl}(1|n)$ representations. For this purpose, it is convenient to take

$$\{E_{00}, E_{n0}, E_{0n}, E_{nn}\} \tag{4.4}$$

as the basis elements of $\mathfrak{gl}(1|1)$, and $\{E_{ij}|1 \leqslant i, j \leqslant n-1\}$ as the basis elements of $\mathfrak{gl}(n-1)$. Indeed, with this choice the actions of the $\mathfrak{gl}(1|1)$ generators on $|m\rangle_E$ only change the labels in the second row of the GZ pattern (see (A.5)–(A.6)), and the actions of the $\mathfrak{gl}(n-1)$ generators only change the last $(n-1)$ rows (see (A.3)–(A.4) for $2 \leqslant k \leqslant n-1$). Otherwise said, the last $(n-1)$ rows of $|m\rangle_E$ coincide with the usual $\mathfrak{gl}(n-1)$ GZ-basis labels. Note that the action of the diagonal elements of $\mathfrak{gl}(1|1)$ is given by

$$E_{00}|m\rangle_E = a|m\rangle_E, \qquad E_{nn}|m\rangle_E = b|m\rangle_E, \tag{4.5}$$

with

$$a = \sum_{j=0}^{n} m_{j,n+1} - \sum_{j=1}^{n} m_{jn} = m_{0,n+1} - \sum_{j=1}^{n} \theta_j, \tag{4.6}$$

$$b = \sum_{j=1}^{n} m_{jn} - \sum_{j=1}^{n-1} m_{j,n-1}. \tag{4.7}$$

For a given unitary representation $W([m]_{n+1})$ of $\mathfrak{gl}(1|n)$, the decomposition to $\mathfrak{gl}(1|1) \oplus \mathfrak{gl}(n-1)$ is thus completely determined by listing all possible rows $[m]_n$ and $[m]_{n-1}$, i.e.

$$\begin{aligned} m_{1n} = m_{1,n+1} + \theta_1, \qquad &\cdots m_{n-1,n} = m_{n-1,n+1} + \theta_{n-1}, \quad m_{nn} = m_{n,n+1} + \theta_n \\ m_{1,n-1}, \qquad &\cdots m_{n-1,n-1} \end{aligned} \tag{4.8}$$

subject to conditions (GZ3) and (GZ4), with $\theta_i \in \{0, 1\}$.

Let us investigate the $\mathfrak{gl}(1|1)$ weight (a, b) of a vector $|m\rangle_E$ more carefully. For a *typical* unitary representation $W([m]_{n+1})$, one finds

$$\begin{aligned} a + b &= \sum_{j=0}^{n} m_{j,n+1} - \sum_{j=1}^{n-1} m_{j,n-1} = m_{0,n+1} + m_{n,n+1} + \sum_{j=1}^{n-1}(m_{j,n+1} - m_{j,n-1}) \\ &= m_{0,n+1} + m_{n,n+1} + \sum_{j=1}^{n-1}(m_{jn} - m_{j,n-1}) - \sum_{j=1}^{n-1} \theta_j. \end{aligned}$$

But by (GZ4) $m_{jn} - m_{j,n-1} \geqslant 0$, and by (2.19) $m_{0,n+1} + m_{n,n+1} > n - 1$, hence

$$a + b > n - 1 - \sum_{j=1}^{n-1} \theta_j \geqslant 0,$$

so $a + b > 0$ and (a, b) can be the weight of a typical two-dimensional $\mathfrak{gl}(1|1)$ representation only.

For an *atypical* unitary representation $W([m]_{n+1})$, satisfying (2.20), (GZ2)–(GZ4) imply that $\theta_k = \theta_{k+1} = \cdots = \theta_n = 0$. Then

$$\begin{aligned} a + b &= m_{0,n+1} + m_{n,n+1} + \sum_{j=1}^{n-1}(m_{jn} - m_{j,n-1}) - \sum_{j=1}^{n-1} \theta_j \\ &= m_{0,n+1} + m_{k,n+1} + \sum_{j=1}^{n-1}(m_{jn} - m_{j,n-1}) - \sum_{j=1}^{k-1} \theta_j \\ &= (k-1) - \sum_{j=1}^{k-1} \theta_j + \sum_{j=1}^{n-1}(m_{jn} - m_{j,n-1}). \end{aligned}$$

So $a + b \geqslant 0$, and $a + b$ can be equal to zero if and only if

$$\theta_1 = \theta_2 = \cdots = \theta_{k-1} = 1, \qquad \text{and} \quad m_{j,n-1} = m_{jn} \quad \text{for all} \quad j = 1, 2, \ldots, n-1. \quad (4.9)$$

Consequently, in the $\mathfrak{gl}(1|n) \to \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(n-1)$ decomposition

$$W([m]_{n+1}) \to \bigoplus W([a,b]) \times V([m]_{n-1}), \qquad (4.10)$$

the representations $W([a,b])$ are always typical if $W([m]_{n+1})$ is typical. If $W([m]_{n+1})$ is atypical of type k, again all $W([a,b])$ are typical, except for one single component where $a + b = 0$ and where the labels of the $\mathfrak{gl}(n-1)$ representation $V([m]_{n-1})$ are given by

$$\begin{aligned} m_{j,n-1} = m_{jn} = m_{j,n+1} + 1 \qquad & \text{for} \quad j = 1, \ldots, k-1; \\ m_{j,n-1} = m_{jn} = m_{j,n+1} \qquad & \text{for} \quad j = k, \ldots, n-1. \end{aligned} \qquad (4.11)$$

Following (4.8), for a fixed $(n+1)$-tuple $[m]_{n+1}$ satisfying (2.15), the set of $(n-1)$-tuples appearing in the decomposition to $\mathfrak{gl}(n-1)$ is given by

$$\mathcal{M}([m]_{n+1}) = \{[m]_{n-1} | m_{i,n+1} + 1 - m_{i,n-1}, \qquad m_{i,n-1} - m_{i+1,n+1} \in \mathbb{Z}_+, \qquad (4.12)$$
$$(i = 1, \ldots, n-1); \qquad m_{i,n-1} - m_{i+1,n-1} \in \mathbb{Z}_+, \qquad (i = 1, \ldots, n-2)\}.$$

To see how often such a $\mathfrak{gl}(n-1)$ representation labelled by $[m]_{n-1}$ appears in the decomposition of $W([m]_{n+1})$, one should count the number of allowed θ_i s in (4.8). Since each $\theta_i \in \{0, 1\}$, this number will be a power of 2. Whether both values for θ_i are allowed depends not only on the (a)typicality of $W([m]_{n+1})$, but also on the tuples $[m]_{n+1}$ and $[m]_{n-1}$ themselves (whether some consecutive numbers are equal, whether some $m_{i,n-1}$ is equal to $m_{i,n+1} + 1$, etc). Both values for θ_i are allowed if $m_{i-1,n-1} - m_{i,n+1} - 1 \in \mathbb{Z}_+$ and $m_{i,n+1} - m_{i,n-1} \in \mathbb{Z}_+$ (for $i = 1$, the first condition disappears since $m_{0,n-1}$ is not a GZ label, and for $i = n$ the second condition disappears since $m_{n,n-1}$ is not a GZ label). So let us consider

$$T(m_{i-1,n-1} - m_{i,n+1} - 1 \in \mathbb{Z}_+ \qquad \text{and} \quad m_{i,n+1} - m_{i,n-1} \in \mathbb{Z}_+), \qquad (4.13)$$

where $T(A) = 1$ if A is true and $T(A) = 0$ if A is false, and

$$N([m]_{n+1}, [m]_{n-1}) = \sum_{i=1}^{n} T(m_{i-1,n-1} - m_{i,n+1} - 1 \in \mathbb{Z}_+ \quad \text{and} \quad m_{i,n+1} - m_{i,n-1} \in \mathbb{Z}_+). \qquad (4.14)$$

Then, for a typical $\mathfrak{gl}(1|n)$ representation $W([m]_{n+1})$, the number of $\mathfrak{gl}(n-1)$ representations $V([m]_{n-1})$ appearing in the decomposition (with $[m]_{n-1} \in \mathcal{M}([m]_{n+1})$) is given by

$$2^{N([m]_{n+1}, [m]_{n-1})}. \qquad (4.15)$$

For a $\mathfrak{gl}(1|n)$ representation that is atypical of type k, the result is essentially the same but now all $\theta_k = \cdots = \theta_n = 0$. So in this case the result is still given by (4.15), except that the upper bound of the sum in (4.14) is $k - 1$ instead of n. It will be convenient to have a notation for the set of allowed n-tuples, for a given $(n+1)$-tuple $[m]_{n+1}$ and a given $(n-1)$-tuple $[m]_{n-1}$:

$$\mathcal{A}([m]_{n+1}, [m]_{n-1}) = \{[m]_n | [m]_{n+1}, [m]_n \quad \text{and} \quad [m]_{n-1} \quad \text{satisfy (GZ2)–(GZ4)}\}. \qquad (4.16)$$

So the number of elements of $\mathcal{A}([m]_{n+1}, [m]_{n-1})$ is given by (4.15).

Knowing the multiplicity of $V([m]_{n-1})$, one can now determine the $\mathfrak{gl}(1|1)$ weights (a, b) for each appearance of $V([m]_{n-1})$ in the decomposition of $W([m]_{n+1})$, and collect these according to irreducible representations of $\mathfrak{gl}(1|1)$ (which are one or two dimensional). This gives rise to the following:

$$W([m]_{n+1}) \to \bigoplus_{[m]_{n-1} \in \mathcal{M}([m]_{n+1})} \left(\bigoplus_{i=0}^{N-1} \binom{N-1}{i} W([a-i, b+i]) \right) \times V([m]_{n-1}), \qquad (4.17)$$

where

$$N \equiv N([m]_{n+1}, [m]_{n-1}),$$

$$a = \sum_{j=0}^{n} m_{j,n+1} - \min_{[m]_n \in \mathcal{A}([m]_{n+1}, [m]_{n-1})} \left(\sum_{j=1}^{n} m_{jn} \right),$$

$$b = -a + \sum_{j=0}^{n} m_{j,n+1} - \sum_{j=1}^{n-1} m_{j,n-1}.$$

Note that for typical representations each $N > 0$. For representations atypical of type k, there is one single $(n-1)$-tuple $[m]_{n-1}$ for which $N = N([m]_{n+1}, [m]_{n-1}) = 0$, namely the case (4.11). For this $(n-1)$-tuple, the term on the right-hand side of (4.17) should be replaced by

$$W([a, -a]) \times V([m]_{n-1}). \tag{4.18}$$

It will be important to note that the range of values for $a + b$ in $W([m]_{n+1})$ goes in steps of 1 and follows from (4.17); it is given by

$$m_{0,n+1} + m_{1,n+1}, \qquad m_{0,n+1} + m_{1,n+1} - 1, \ldots, m_{0,n+1} + m_{n,n+1} - n + 1 (> 0) \tag{4.19}$$

for typical representations, and by

$$m_{0,n+1} + m_{1,n+1}, \qquad m_{0,n+1} + m_{1,n+1} - 1, \ldots, m_{0,n+1} + m_{k,n+1} - k + 1 (= 0) \tag{4.20}$$

for representations atypical of type k.

We are now in a position to solve the eigenvalue problem for $\hat{q}_r$. Remember that $\hat{q}_r = \sqrt{\frac{\hbar\gamma}{\mu n}}(E_{0n} + E_{n0})$, see (2.11). Hence in a two-dimensional typical $\mathfrak{gl}(1|1)$ representation $W([a, b])$ $(a + b > 0)$, it follows from (4.2) and (4.4) that the eigenvalues of $E_{0n} + E_{n0}$ are $\pm\sqrt{a + b}$, whereas in a one-dimensional atypical $\mathfrak{gl}(1|1)$ representation $W([a, b])$ $(a+b = 0)$, the eigenvalue is 0.

So we find the following result.

Theorem 3. *Let $W([m]_{n+1})$ be a unitary representation of $\mathfrak{gl}(1|n)$.*

(a) If $W([m]_{n+1})$ is typical, the eigenvalues of $\hat{q}_r$ are given by $\pm\sqrt{\frac{\hbar\gamma K}{\mu n}}$ where the range of K, in steps of 1, is determined by

$$K = m_{0,n+1} + m_{1,n+1}, \qquad m_{0,n+1} + m_{1,n+1} - 1, \ldots, m_{0,n+1} + m_{n,n+1} - n + 1. \tag{4.21}$$

The multiplicity of each eigenvalue $\pm\sqrt{\frac{\hbar\gamma K}{\mu n}}$ is determined by (4.17) and is of the form

$$\sum 2^N \dim(V([m]_{n-1})), \tag{4.22}$$

where the sum is over all $(n-1)$-tuples $[m]_{n-1}$ from $\mathcal{M}([m]_{n+1})$ for which $\sum_{j=0}^{n} m_{j,n+1} - \sum_{j=1}^{n-1} m_{j,n-1} = K$. The dimensions of $\mathfrak{gl}(n-1)$ representations $V([m]_{n-1})$ are well known [19, p 33].

(b) If $W([m]_{n+1})$ is atypical of type k, the eigenvalues of $\hat{q}_r$ are given by $\pm\sqrt{\frac{\hbar\gamma K}{\mu n}}$, where $K = 0, 1, 2, \ldots, m_{0,n+1} + m_{1,n+1}$. The multiplicity of each nonzero eigenvalue is again determined by (4.17) and given by a formula similar to (4.22). The multiplicity of the zero eigenvalue is $\dim V([m]_{n-1})$, with $[m]_{n-1}$ given by (4.11).

5. Relation between the two GZ-basis vectors

Consider the unitary $\mathfrak{gl}(1|n)$ representation $W([m]_{n+1})$. On the one hand, $W([m]_{n+1})$ has a GZ basis of vectors $|m)_e$, with the standard action of e_{ij} on these vectors determined by (A.1)–(A.8). The highest weight vector $|\Lambda)_e$ with respect to this $\mathfrak{gl}(1|n)$ basis is given by (2.21). Note that the highest weight vector is uniquely characterized by

$$e_{j,j+1}|\Lambda)_e = 0 \qquad (1 \leqslant j \leqslant n-1), \tag{5.1}$$

$$e_{0j}|\Lambda)_e = 0 \qquad (1 \leqslant j \leqslant n). \tag{5.2}$$

The last condition is guaranteed by the fact that for $|\Lambda)_e$ all $\theta_i = 0$ in (GZ2). The first condition follows from action (A.3).

On the other hand, we have considered a new basis E_{ij} for $\mathfrak{gl}(1|n)$, determined by (3.6)–(3.9). With respect to this new basis, $W([m]_{n+1})$ has a new GZ basis with vectors $|m)_E$, and a new highest weight vector $|\Lambda)_E$. We want to find an expression for $|\Lambda)_E$ as a linear combination of vectors $|m)_e$:

$$|\Lambda)_E = \sum c_m |m)_e. \tag{5.3}$$

So, we should require

$$E_{j,j+1}|\Lambda)_E = 0 \qquad (1 \leqslant j \leqslant n-1), \tag{5.4}$$

$$E_{0j}|\Lambda)_E = 0 \qquad (1 \leqslant j \leqslant n). \tag{5.5}$$

Since each E_{0j} is a linear combination of elements e_{0l}, it follows that the linear combination in (5.3) consists of m-patterns with all $\theta_i = 0$ in (GZ2). So we should examine the elements $E_{j,j+1}$ more closely, and in particular their action on vectors $|m)_e$.

We can compute $E_{j,j+1}$ by means of (3.8)–(3.9) and $E_{j,j+1} = \{E_{j0}, E_{0,j+1}\}$. For $1 \leqslant j \leqslant n-2$, this gives

$$E_{j,j+1} = \gamma_{j+1}\gamma_{j+2}\sqrt{\frac{\gamma_1^2 + \cdots + \gamma_j^2}{\gamma_1^2 + \cdots + \gamma_{j+2}^2}}\left(\sum_{l_1=1}^{j}\sum_{l_2=1}^{j+1}\frac{e^{2\pi i r(l_1-l_2)/n}\gamma_{l_1}\gamma_{l_2}}{(\gamma_1^2 + \cdots + \gamma_j^2)(\gamma_1^2 + \cdots + \gamma_{j+1}^2)}e_{l_1 l_2}\right.$$
$$\left. -\sum_{l_1=1}^{j}\frac{e^{2\pi i r(l_1-j-2)/n}\gamma_{l_1}}{(\gamma_1^2 + \cdots + \gamma_j^2)\gamma_{j+2}}e_{l_1,j+2} - \sum_{l_2=1}^{j+1}\frac{e^{2\pi i r(j+1-l_2)/n}\gamma_{l_2}}{(\gamma_1^2 + \cdots + \gamma_{j+1}^2)\gamma_{j+1}}e_{j+1,l_2} + \frac{e^{-2\pi i r/n}}{\gamma_{j+1}\gamma_{j+2}}e_{j+1,j+2}\right), \tag{5.6}$$

and for $j = n-1$:

$$E_{n-1,n} = \frac{\gamma_n\sqrt{\gamma_1^2 + \cdots + \gamma_{n-1}^2}}{\gamma_1^2 + \cdots + \gamma_n^2}\left(\sum_{l_1=1}^{n-1}\sum_{l_2=1}^{n}\frac{e^{2\pi i r(l_1-l_2)/n}\gamma_{l_1}\gamma_{l_2}}{(\gamma_1^2 + \cdots + \gamma_{n-1}^2)}e_{l_1 l_2} - \sum_{l=1}^{n}\frac{e^{-2\pi i r l/n}\gamma_l}{\gamma_n}e_{nl}\right). \tag{5.7}$$

The following type of vectors from $W([m]_{n+1})$ will play an essential role:

$$|m(d))_e = \begin{vmatrix} m_{0,n+1} & m_{1,n+1} & m_{2,n+1} & \cdots & & \cdots & m_{n-2,n+1} & m_{n-1,n+1} & m_{n,n+1} \\ & m_{1,n+1} & m_{2,n+1} & \cdots & & \cdots & m_{n-2,n+1} & m_{n-1,n+1} & m_{n,n+1} \\ & m_{1,n+1} & m_{2,n+1} & \cdots & & \cdots & m_{n-2,n+1} & m_{n-1,n-1} & \\ & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ & m_{1,n+1} & m_{2,n+1} & m_{3,n+1} & m_{44} & & & & \\ & m_{1,n+1} & m_{2,n+1} & m_{33} & & & & & \\ & m_{1,n+1} & m_{22} & & & & & & \\ & m_{11} & & & & & & & \end{vmatrix}_e. \tag{5.8}$$

So in this expression, all labels in the GZ pattern are fixed, except the $(n-1)$ bottom labels $d = (m_{11}, m_{22}, \ldots, m_{n-1,n-1})$ which are allowed to vary according to (GZ4).

Now we have the following result.

Proposition 4. *The highest weight vector of $W([m]_{n+1})$ according to the new $\mathfrak{gl}(1|n)$ basis E_{ij} is given by*

$$|\Lambda\rangle_E = \frac{1}{\sqrt{\mathcal{N}}} \sum_{m_{n-1,n-1}=m_{n,n+1}}^{m_{n-1,n+1}} \sum_{m_{n-2,n-2}=m_{n-1,n-1}}^{m_{n-2,n+1}} \cdots \sum_{m_{22}=m_{33}}^{m_{2,n+1}} \sum_{m_{11}=m_{22}}^{m_{1,n+1}} (-1)^{m_{11}+\cdots+m_{n-1,n-1}}$$

$$\times e^{-2\pi i r(m_{11}+\cdots+m_{n-1,n-1})/n} \left[\binom{m_{1,n+1}-m_{22}}{m_{1,n+1}-m_{11}} \binom{m_{2,n+1}-m_{33}}{m_{2,n+1}-m_{22}} \cdots \right.$$

$$\times \cdots \left. \binom{m_{n-2,n+1}-m_{n-1,n-1}}{m_{n-2,n+1}-m_{n-2,n-2}} \binom{m_{n-1,n+1}-m_{n,n+1}}{m_{n-1,n+1}-m_{n-1,n-1}} \right]^{1/2}$$

$$\times \gamma_1^{m_{1,n+1}-m_{11}} \gamma_2^{m_{11}-m_{22}} \gamma_3^{m_{22}-m_{33}} \cdots \gamma_{n-1}^{m_{n-2,n-2}-m_{n-1,n-1}} \gamma_n^{m_{n-1,n-1}-m_{n,n+1}} |m(d)\rangle_e, \tag{5.9}$$

where $\mathcal{N}$ is a normalization factor given by

$$\mathcal{N} = \left(\gamma_1^2+\gamma_2^2\right)^{m_{1,n+1}-m_{2,n+1}} \left(\gamma_1^2+\gamma_2^2+\gamma_3^2\right)^{m_{2,n+1}-m_{3,n+1}} \cdots \left(\gamma_1^2+\cdots+\gamma_n^2\right)^{m_{n-1,n+1}-m_{n,n+1}}. \tag{5.10}$$

Proof. We shall only give a sketch of the proof, which requires careful computations. Essentially, one considers for $1 \leqslant j \leqslant n-1$ the action $E_{j,j+1}|\Lambda\rangle_E$, using (5.6)–(5.7), (5.9) and the explicit action on the GZ basis given by (A.2)–(A.4). In the resulting expression, one combines all contributions with the same GZ pattern and verifies that the coefficients become zero. In this computation, it is essential to know the action of an element $e_{l_1 l_2}$ on vectors of the form (5.8). From the general action (A.2)–(A.4), one deduces

- if $l_1 = l_2$, then $e_{l_1 l_2}|m(d)\rangle_e$ gives just a constant times $|m(d)\rangle_e$;
- if $l_1 < l_2$, then $e_{l_1 l_2}|m(d)\rangle_e$ gives only one term with a vector which is again of the form (5.8), and
- if $l_1 > l_2$, then $e_{l_1 l_2}|m(d)\rangle_e$ gives a linear combination of several vectors. Some of these vectors are of the form (5.8). The other vectors are not of the form (5.8): they have the same labels as $|m(d)\rangle_e$, but with one of the labels in row l decreased by 1, for every $l = l_1 - 1, l_1 - 2, \ldots, l_2$.

A careful examination shows that taking together all contributions to vectors that are not of type (5.8) in the expansion of $E_{j,j+1}|\Lambda\rangle_E$ gives zero. So it remains to compute the coefficients of vectors of the type (5.8) in the expansion of $E_{j,j+1}|\Lambda\rangle_E$ ($j = 1, \ldots, n-1$). Explicitly, this gives rise to a coefficient of the form

$$\frac{\prod_{i=1}^{j}(m_{i,n+1}-m_{ii})}{\prod_{i=1}^{j-1}(m_{i,n+1}-m_{i+1,i+1})} + \sum_{l=1}^{j-1}(m_{ll}-m_{l+1,l+1})\frac{\prod_{i=l+1}^{j}(m_{i,n+1}-m_{ii})}{\prod_{i=l}^{j-1}(m_{i,n+1}-m_{i+1,i+1})} - m_{j,n+1} + m_{jj}. \tag{5.11}$$

Denote $m_{i,n+1} = x_i$ and $m_{ii} = y_i$. We shall prove that

$$\frac{\prod_{i=1}^{j}(x_i-y_i)}{\prod_{i=1}^{j-1}(x_i-y_{i+1})} + \sum_{l=1}^{j-1}(y_l-y_{l+1})\frac{\prod_{i=l+1}^{j}(x_i-y_i)}{\prod_{i=l}^{j-1}(x_i-y_{i+1})} = x_j - y_j \tag{5.12}$$

for arbitrary variables x_i and y_i, implying that (5.11) is indeed always zero. Identity (5.12) is true for $j = 1$. Suppose it is true for a fixed j, and let us consider it for $j + 1$:

$$\frac{\prod_{i=1}^{j+1}(x_i-y_i)}{\prod_{i=1}^{j}(x_i-y_{i+1})} + \sum_{l=1}^{j}(y_l-y_{l+1})\frac{\prod_{i=l+1}^{j+1}(x_i-y_i)}{\prod_{i=l}^{j}(x_i-y_{i+1})} = x_{j+1} - y_{j+1}. \tag{5.13}$$

The left-hand side of (5.13) yields, using (5.12) and induction on j:

$$\frac{(x_{j+1} - y_{j+1})}{(x_j - y_{j+1})} \left(\frac{\prod_{i=1}^{j}(x_i - y_i)}{\prod_{i=1}^{j-1}(x_i - y_{i+1})} + \sum_{l=1}^{j-1}(y_l - y_{l+1}) \frac{\prod_{i=l+1}^{j}(x_i - y_i)}{\prod_{i=l}^{j-1}(x_i - y_{i+1})} \right)$$

$$+ (y_j - y_{j+1}) \frac{(x_{j+1} - y_{j+1})}{(x_j - y_{j+1})} = \frac{(x_{j+1} - y_{j+1})}{(x_j - y_{j+1})}(x_j - y_j)$$

$$+ (y_j - y_{j+1}) \frac{(x_{j+1} - y_{j+1})}{(x_j - y_{j+1})} = x_{j+1} - y_{j+1}.$$

So the identity holds in general. This shows that all coefficients in the expansion of $E_{j,j+1}|\Lambda\rangle_E$ are zero, in other words $E_{j,j+1}|\Lambda\rangle_E = 0$.

To see that $\mathcal{N}$ gives the right normalization coefficient, one can simply expand the right-hand side of (5.10). This gives, after appropriate relabelling of the summation indices:

$$\sum_{k_{n-1}=m_{n,n+1}}^{m_{n-1,n+1}} \sum_{k_{n-2}=k_{n-1}}^{m_{n-2,n+1}} \cdots \sum_{k_2=k_3}^{m_{2,n+1}} \sum_{k_1=k_2}^{m_{1,n+1}} \binom{m_{1,n+1} - k_2}{m_{1,n+1} - k_1} \binom{m_{2,n+1} - k_3}{m_{2,n+1} - k_2} \cdots \tag{5.14}$$

$$\cdots \binom{m_{n-1,n+1} - m_{n,n+1}}{m_{n-1,n+1} - k_{n-1}} \gamma_1^{2(m_{1,n+1} - k_1)} \gamma_2^{2(k_1 - k_2)} \gamma_3^{2(k_2 - k_3)} \cdots \gamma_n^{2(k_{n-1} - m_{n,n+1})}. \tag{5.15}$$

Clearly, this is just the norm of the vector given as a summand on the right-hand side of (5.9).

In principle, we now have a solution to our eigenvector problem, i.e. we can give a set of orthonormal eigenvectors of $\hat{q}_r$ for $W([m]_{n+1})$ in terms of the basis $|m\rangle_e$. First of all, (4.17) gives the decomposition of $W([m]_{n+1})$ with respect to $\mathfrak{gl}(1|1) \oplus \mathfrak{gl}(n-1)$, so from this step one can express the weight vectors v and w of every $W([a, b])$ $(a + b > 0)$ in terms of vectors $|m\rangle_E$. Then (4.2) and (2.11) imply that the eigenvectors of $\hat{q}_r$ are $(v \pm w)/\sqrt{2}$:

$$\hat{q}_r \frac{v \pm w}{\sqrt{2}} = \pm \sqrt{\frac{\hbar\gamma}{\mu n}} \sqrt{a + b} \frac{v \pm w}{\sqrt{2}}. \tag{5.16}$$

But in principle every $|m\rangle_E$, and thus also v and w, can be expressed as powers of E_{ij} $(i > j)$ acting on $|\Lambda\rangle_E$ (in practice this can be hard, though). The rest is now routine: write every such E_{ij} in terms of e_{ij}, and use (5.9). This leads to an expression of the eigenvectors in terms of the basis $|m\rangle_e$.

In the following sections, we shall illustrate how this works for two special types of unitary representations.

6. The Fock representations $W([p, 0, \ldots, 0]) \equiv W(p)$

One interesting class of representations [7] of $\mathfrak{gl}(1|n)$ is that with $[m]_{n+1} = [p, 0, \ldots, 0]$, i.e. with highest weight $\Lambda = p\epsilon$. The representation space $W([p, 0, \ldots, 0])$ is simply denoted by $W(p)$. It follows from (U1)-(U2) that $W(p)$ is unitary when either $p > n - 1$ (typical case) or else $p = 0, 1, \ldots, n - 1$ (atypical of type $p + 1$). In the notation of (2.17), the GZ patterns of $W(p)$ consist of zeros and ones only (apart from the label p), so it will be convenient to use a simpler notation for these vectors. The GZ-basis vectors of $W(p)$ will simply be denoted by $w(\varphi_1, \ldots, \varphi_n) \equiv w(\varphi)$, where the relation to the GZ labels is determined by [6]

$$\varphi_i = \sum_{j=1}^{i} m_{ji} - \sum_{j=1}^{i-1} m_{j,i-1}. \tag{6.1}$$

The constraints (GZ2)–(GZ4) for the GZ labels lead to $\varphi_i \in \{0, 1\}$ and $\sum_{i=1}^{n} \varphi_i \leqslant \min(p, n)$. The representations $W(p)$ and the basis vectors $w(\varphi_1, \ldots, \varphi_n)$ have been constructed by means of Fock space techniques, and the action of the $\mathfrak{gl}(1|n)$ generators is very simple, see [7]. The Fock construction gives all vectors in terms of the highest weight vector $|\Lambda\rangle_e \equiv w(0, \ldots, 0)$:

$$w(\varphi) = w(\varphi_1, \ldots, \varphi_n) = \frac{e_{10}^{\varphi_1} e_{20}^{\varphi_2} \cdots e_{n0}^{\varphi_n}}{\sqrt{p(p-1)\cdots(p-|\varphi|+1)}} w(0, \ldots, 0),$$

where $|\varphi| = \sum \varphi_i$. The action of the e_{ij} on such vectors is determined by ($1 \leqslant k \leqslant n$):

$$e_{00} w(\varphi) = (p - |\varphi|) w(\varphi), \tag{6.2}$$

$$e_{kk} w(\varphi) = \varphi_k w(\varphi), \tag{6.3}$$

$$e_{k0} w(\varphi) = (1 - \varphi_k)(-1)^{\varphi_1 + \cdots + \varphi_{k-1}} \sqrt{p - |\varphi|}\, w(\varphi_1, \ldots, \varphi_k + 1, \ldots, \varphi_n), \tag{6.4}$$

$$e_{0k} w(\varphi) = \varphi_k (-1)^{\varphi_1 + \cdots + \varphi_{k-1}} \sqrt{p - |\varphi| + 1}\, w(\varphi_1, \ldots, \varphi_k - 1, \ldots, \varphi_n). \tag{6.5}$$

Now we introduce the second GZ basis, denoted by $|m\rangle_E$ in the previous paragraphs. Analogous to the previous basis, we shall use a simpler notation, namely $v(\phi) = v(\phi_1, \ldots, \phi_n)$, with each $\phi_i \in \{0, 1\}$. This basis is defined by

$$v(\phi) = v(\phi_1, \ldots, \phi_n) \equiv \frac{E_{10}^{\phi_1} E_{20}^{\phi_2} \cdots E_{n0}^{\phi_n}}{\sqrt{p(p-1)\cdots(p-|\phi|+1)}} v(0, \ldots, 0), \tag{6.6}$$

where E_{j0} is determined by (3.6) and (3.8), and $v(0, \ldots, 0)$ is the highest weight vector $|\Lambda\rangle_E$ with respect to the E_{ij} basis of $\mathfrak{gl}(1|n)$. This vector is given by (5.9). However, in the current case there is only one vector of the type (5.8), so $|\Lambda\rangle_E = |\Lambda\rangle_e$, in other words $v(0, \ldots, 0) = w(0, \ldots, 0)$. This implies, in particular, that $v(\phi)$ is a linear combination of vectors $w(\varphi)$ with $|\varphi| = |\phi|$.

For the typical case, all N-values in (4.17) are 1, and the decomposition becomes

$$W(p) \to \bigoplus_{K=0}^{n-1} W([p - K, 0]) \times V([\underbrace{1, \ldots, 1}_{K}, 0, \ldots, 0]), \tag{6.7}$$

where the $\mathfrak{gl}(n-1)$ representation $V([1, \ldots, 1, 0, \ldots, 0])$ has K ones and $n - 1 - K$ zeros, with $\dim V([1, \ldots, 1, 0, \ldots, 0]) = \binom{n-1}{K}$. Consequently, $\hat{q}_r$ has $2n$ eigenvalues $\pm x_K = \pm\sqrt{\frac{\hbar\gamma}{\mu n}(p - K)}$, where $0 \leqslant K \leqslant n - 1$, with multiplicities $\binom{n-1}{K}$. The orthonormal eigenvectors are

$$\psi_{r, \pm x_K, \phi} = \frac{1}{\sqrt{2}} v(\phi_1, \ldots, \phi_{n-1}, 0) \pm \frac{(-1)^{\phi_1 + \cdots + \phi_{n-1}}}{\sqrt{2}} v(\phi_1, \ldots, \phi_{n-1}, 1), \tag{6.8}$$

where $\phi_1 + \cdots + \phi_{n-1} = K$. It is indeed easy to check, with $\hat{q}_r = \sqrt{\frac{\hbar\gamma}{\mu n}}(E_{0n} + E_{n0})$, that

$$\hat{q}_r \psi_{r, \pm x_K, \phi} = \pm\sqrt{\frac{\hbar\gamma(p - K)}{\mu n}}\, \psi_{r, \pm x_K, \phi}. \tag{6.9}$$

Thus we have the following.

Proposition 5. *In the typical representation $W(p) = W([p, 0, \ldots, 0])$ ($p > n - 1$), the operator $\hat{q}_r$ has $2n$ distinct eigenvalues given by $\pm x_K = \pm\sqrt{\frac{\hbar\gamma}{\mu n}(p - K)}$, where $0 \leqslant K \leqslant n - 1$. The multiplicity of the eigenvalue $\pm x_K$ is $\binom{n-1}{K}$. The eigenvectors of $\hat{q}_r$ for the eigenvalue $\pm x_K$ contain, when expanded in the standard basis $w(\varphi)$, only vectors with $|\varphi| = K$ or $|\varphi| = K + 1$. A set of orthonormal eigenvectors is given by (6.8).*

What happens in the atypical case? Then $p \in \{0, 1, \ldots, n-1\}$ and $W(p)$ is atypical of type $p + 1$. Now decomposition (4.17) becomes

$$W(p) \to \bigoplus_{K=0}^{p} W([p - K, 0]) \times V([\underbrace{1, \ldots, 1}_{K}, 0, \ldots, 0]). \tag{6.10}$$

Consequently, $\hat{q}_r$ has $2p$ nonzero eigenvalues $\pm x_K = \pm\sqrt{\frac{\hbar\gamma}{\mu n}(p - K)}$, where $0 \leqslant K \leqslant p-1$, with multiplicities $\binom{n-1}{K}$, and one zero eigenvalue $x_p = 0$ with multiplicity $\binom{n-1}{p}$. For a nonzero eigenvalue, the orthonormal eigenvectors take the same form as (6.8). For the zero eigenvalue, the orthonormal eigenvectors are simply all vectors $v(\phi)$ with $|\phi| = p$ and $\phi_n = 0$.

Note that the spectrum of $\hat{q}_r$ is independent of r, i.e. independent of the location of the oscillator in the linear chain of n oscillators. The eigenvectors, however, do depend on r. This is because in (6.6) the generators E_{j0} do indeed depend on r, see (3.6) and (3.8).

Using (6.8), (6.6), (3.6) and (3.8), one can explicitly compute the coefficients

$$\psi_{r, \pm x_{|\phi|}, \phi} = \sum_{\varphi} C^{\varphi}_{r, \pm x_{|\phi|}, \phi} w(\varphi) \tag{6.11}$$

for the expansion of the $\hat{q}_r$ eigenvectors in terms of the stationary states $w(\varphi)$. We have already noted that on the right-hand side of (6.11), only terms with $|\varphi| = |\phi|$ or $|\varphi| = |\phi| + 1$ can be nonzero.

When the quantum system is in a fixed eigenstate $w(\varphi)$ of $\hat{H}$, then the probability of measuring for $\hat{q}_r$ the eigenvalue $\pm x_K$ is given by

$$P(\varphi, r, \pm x_K) = \sum_{\phi, \phi_1 + \phi_2 + \cdots + \phi_{n-1} = K} |C^{\varphi}_{r, \pm x_K, \phi}|^2. \tag{6.12}$$

Without giving details of the computations, we have deduced

$$P(\varphi, r, x_K) = \begin{cases} \dfrac{1}{2\gamma} \displaystyle\sum_{j=1}^{n} (1 - \varphi_j)\gamma_j^2 & \text{when} \quad |\varphi| = K \\[4mm] \dfrac{1}{2\gamma} \displaystyle\sum_{j=1}^{n} \varphi_j \gamma_j^2 & \text{when} \quad |\varphi| = K + 1, \\[4mm] 0 & \text{otherwise.} \end{cases} \tag{6.13}$$

Since

$$P(\varphi, r, -x_K) = P(\varphi, r, x_K),$$

one has

$$P(\varphi, r, x_{|\varphi|}) + P(\varphi, r, x_{|\varphi|-1}) + P(\varphi, r, -x_{|\varphi|}) + P(\varphi, r, -x_{|\varphi|-1}) = 1,$$

implying the following: when the quantum system is in the stationary state $w(\varphi)$, a measurement of $\hat{q}_r$ leads to four possible values $\pm x_{|\varphi|}, \pm x_{|\varphi|-1}$, with probabilities given by (6.12) and (6.13).

7. The ladder representations $W([1, p - 1, 0, \ldots, 0]) \equiv V(p)$

Another interesting class of representations [6] of $\mathfrak{gl}(1|n)$ is that with $[m]_{n+1} = [1, p - 1, 0, \ldots, 0]$, denoted by $V(p)$. By (GZ1), p is a positive integer, and by (U2) it is a unitary representation atypical of type 2. Notation (2.17) for the GZ patterns of $V(p)$ has again too many zeros to be convenient, so the vectors will be denoted in a simpler way. In this case, one can write the $|m)_e$ s as $w(\theta; s_1, s_2 \ldots, s_n) \equiv w(\theta; s)$, where

$$\theta = p - m_{1n}, \qquad s_1 = m_{11}, \qquad s_k = m_{1k} - m_{1,k-1} \qquad (k = 2, \ldots, n). \tag{7.1}$$

Thus, all vectors of $V(p)$ are described by

$$w(\theta; s) \equiv w(\theta; s_1, s_2, \ldots, s_n), \quad \theta \in \{0, 1\}, \; s_i \in \{0, 1, 2, \ldots\}, \quad \text{and} \; \theta + s_1 + \cdots + s_n = p. \tag{7.2}$$

In this notation the highest weight vector is $|\Lambda\rangle_e = w(1; p - 1, 0, \ldots, 0)$. The action of the $\mathfrak{gl}(1|n)$ generators on the new basis (7.2) is given by $(1 \leqslant k \leqslant n)$:

$$e_{00} w(\theta; s) = \theta w(\theta; s), \tag{7.3}$$

$$e_{kk} w(\theta; s) = s_k w(\theta; s), \tag{7.4}$$

$$e_{k0} w(\theta; s) = \theta \sqrt{s_k + 1} \, w(1 - \theta; s_1, \ldots, s_k + 1, \ldots, s_n), \tag{7.5}$$

$$e_{0k} w(\theta; s) = (1 - \theta) \sqrt{s_k} \, w(1 - \theta; s_1, \ldots, s_k - 1, \ldots, s_n). \tag{7.6}$$

From these one deduces the action of other elements e_{kl}. The ladder representations $V(p)$ and the basis vectors $w(\theta; s_1, s_2 \ldots, s_n)$ can also be constructed by means of negative root vectors acting on the highest weight vector. In particular,

$$w(\theta; s_1, \ldots, s_n) = \frac{e_{n,n-1}^{p-\theta-\sum_{j=1}^{n-1} s_j} e_{n-1,n-2}^{p-\theta-\sum_{j=1}^{n-2} s_j} \cdots e_{32}^{p-\theta-\sum_{j=1}^{2} s_j} e_{21}^{p-\theta-\sum_{j=1}^{1} s_j} e_{10}^{1-\theta}}{\sqrt{p^{1-\theta} \prod_{k=1}^{n-1} \left(p - \theta - \sum_{j=1}^{k} s_j \right)! (s_k + 1)_{p-\theta-\sum_{j=1}^{k} s_j}}}$$

$$w(1; p - 1, 0, \ldots, 0),$$

where $(a)_j = a(a + 1) \cdots (a + j - 1)$ is the Pochhammer symbol or rising factorial.

Now we also introduce the second GZ basis $|m\rangle_E$, but in the same simpler notation, namely $v(\phi; t) = v(\phi; t_1, \ldots, t_n)$, with $\phi \in \{0, 1\}$, $t_i \in \mathbb{Z}_+$ and $\phi + t_1 + \cdots + t_n = p$. This basis is defined by

$$v(\phi; t_1, \ldots, t_n) = \frac{E_{n,n-1}^{p-\phi-\sum_{j=1}^{n-1} t_j} E_{n-1,n-2}^{p-\phi-\sum_{j=1}^{n-2} t_j} \cdots E_{32}^{p-\phi-\sum_{j=1}^{2} t_j} E_{21}^{p-\phi-\sum_{j=1}^{1} t_j} E_{10}^{1-\phi}}{\sqrt{p^{1-\phi} \prod_{k=1}^{n-1} \left(p - \phi - \sum_{j=1}^{k} t_j \right)! (t_k + 1)_{p-\phi-\sum_{j=1}^{k} t_j}}} \tag{7.7}$$

$$v(1; p - 1, 0, \ldots, 0),$$

where E_{j0} is determined by (3.6) and (3.8), $E_{j+1,j} = \{E_{j+1,0}, E_{0,j}\}$, and $v(1; p - 1, 0, \ldots, 0)$ is the highest weight vector $|\Lambda\rangle_E$ with respect to the E_{ij} basis of $\mathfrak{gl}(1|n)$. In general, this vector is given by (5.9), and here this becomes

$$v(1; p - 1, 0, \ldots, 0) = \frac{1}{(\gamma_1^2 + \gamma_2^2)^{(p-1)/2}} \sum_{u=0}^{p-1} (-1)^u \, e^{-2\pi i r u/n} \sqrt{\binom{p-1}{u}}$$

$$\times \gamma_1^{p-1-u} \gamma_2^u w(1; u, p - 1 - u, 0, \ldots, 0). \tag{7.8}$$

Decomposition (4.17) reads

$$V(p) \to W([0, 0]) \times V([p, 0, \ldots, 0]) \oplus \bigoplus_{K=0}^{p-1} W([1, p - 1 - K]) \times V([K, 0, \ldots, 0]), \tag{7.9}$$

where the $\mathfrak{gl}(n - 1)$ representation has dim $V([K, 0, \ldots, 0]) = \binom{n-2+K}{n-2}$. So $\hat{q}_r$ has $2p + 1$ eigenvalues in all, namely $\pm x_K = \pm \sqrt{\frac{\hbar\gamma}{\mu n}(p - K)}$, where $0 \leqslant K \leqslant p - 1$, with multiplicities $\binom{n-2+K}{n-2}$, and $x_p = 0$ with multiplicity $\binom{n-2+p}{n-2}$. The orthonormal eigenvectors for $\pm x_K \neq 0$ are

$$\psi_{r, \pm x_K, t} = \frac{1}{\sqrt{2}} v(1; t_1, \ldots, t_{n-1}, p - 1 - K) \pm \frac{1}{\sqrt{2}} v(0; t_1, \ldots, t_{n-1}, p - K), \tag{7.10}$$

where $t_1 + \cdots + t_{n-1} = K$. For the eigenvalue 0, the eigenvectors read

$$\psi_{r,0,t} = v(0; t_1, \ldots, t_{n-1}, 0), \qquad t_1 + \cdots + t_{n-1} = p. \tag{7.11}$$

In other words

Proposition 6. *In the representation* $V(p) = W([1, p-1, 0, \ldots, 0])$, *the operator* $\hat{q}_r$ *has* $2p + 1$ *distinct eigenvalues given by* $\pm x_K = \pm\sqrt{\frac{\hbar\gamma}{\mu n}(p - K)}$, *where* $0 \leqslant K \leqslant p$. *The multiplicity of the eigenvalue* $\pm x_K$ *is* $\binom{n-2+K}{K}$. *A set of orthonormal eigenvectors is given by (7.10) and (7.11).*

8. Conclusions

In this paper, we managed to determine the eigenvalues of an arbitrary self-adjoint odd element (1.1) of the Lie superalgebra $\mathfrak{gl}(1|n)$ in a unitary representation $W = W([m]_{n+1})$. Furthermore, we gave a construction of a set of orthonormal eigenvectors of this element in W, using the GZ-basis vectors. The problem is of importance in the study of physical properties of the $\mathfrak{gl}(1|n)$ Wigner quantum system solution for a model consisting of a linear chain of n harmonic oscillators coupled by springs, with periodic boundary conditions. In such a description, the position and momentum operators $\hat{q}_r$ and $\hat{p}_r$ of the rth oscillator are such odd elements, see (2.11)–(2.12). We have concentrated on the operator $\hat{q}_r$. Note, by (2.12), that the analysis of $\hat{p}_r$ is very similar: one should replace all constants γ_j by $\sqrt{\beta_j}$, leading to the analogue of (2.11):

$$\hat{p}_r = i\sqrt{\frac{\mu\hbar\beta}{n}}(E_{0n} - E_{n0}), \qquad (\beta = \beta_1 + \cdots + \beta_n). \tag{8.1}$$

Then the counterpart of (5.16) is

$$\hat{p}_r \frac{v \pm iw}{\sqrt{2}} = \mp\sqrt{\frac{\mu\hbar\beta}{n}}\sqrt{a+b}\frac{v \pm iw}{\sqrt{2}}. \tag{8.2}$$

So, up to an overall factor, the spectrum of $\hat{p}_r$ is the same as that of $\hat{q}_r$. The eigenvectors, however, are different, but can be found by a similar construction.

As an application, we have in mind the description of some geometric aspects of the $\mathfrak{gl}(1|n)$ solution of the quantum system described. These aspects depend on the representation considered. For the simple class of Fock representations $W(p)$, some properties were already described in [1]. Clearly, the ladder representations $V(p)$ have a much richer structure. It would be interesting to study such properties for these representations. In particular, we have in mind: position probability distributions for the stationary states $w(\theta; s)$; position probabilities for the other oscillators when one oscillator is in an eigenstate with fixed eigenvalue; average position of the other oscillators when one oscillator is in a fixed position, etc. For all these aspects, one needs the explicit expansion of the orthonormal $\hat{q}_r$ eigenvectors in terms of the basis of stationary states $w(\theta; s)$, as determined in this paper in section 7.

We want to point out that the analysis presented here will be useful not only for the quantum system described here in section 2, but also for the study of related models. For example, a quantum system consisting of a linear chain of harmonic oscillators coupled by springs, but with non-periodic boundary conditions (i.e. with fixed end points) also allows a $\mathfrak{gl}(1|n)$ Wigner quantum system solution. The techniques developed here should be useful in the study of such alternative systems.

Acknowledgments

Appendix

The explicit action of a set of $\mathfrak{gl}(1|n)$ generators on the basis vectors (2.17) was given in [6, equations (2.13)–(2.18)]. For the readability of this paper, we repeat this here. Denote by $|m)_{\pm ij}$ the pattern obtained from $|m)$ by the replacement $m_{ij} \to m_{ij} \pm 1$. Then the action is given by

$$e_{00}|m) = \left(m_{0,n+1} - \sum_{j=1}^{n} \theta_j \right) |m); \tag{A.1}$$

$$e_{kk}|m) = \left(\sum_{j=1}^{k} m_{jk} - \sum_{j=1}^{k-1} m_{j,k-1} \right) |m), \qquad (1 \leqslant k \leqslant n); \tag{A.2}$$

$$e_{k-1,k}|m) = \sum_{j=1}^{k-1} \left(-\frac{\prod_{i=1}^{k}(l_{ik} - l_{j,k-1}) \prod_{i=1}^{k-2}(l_{i,k-2} - l_{j,k-1} - 1)}{\prod_{i\neq j=1}^{k-1}(l_{i,k-1} - l_{j,k-1})(l_{i,k-1} - l_{j,k-1} - 1)} \right)^{1/2} |m)_{+j,k-1},$$
$$(2 \leqslant k \leqslant n); \tag{A.3}$$

$$e_{k,k-1}|m) = \sum_{j=1}^{k-1} \left(-\frac{\prod_{i=1}^{k}(l_{ik} - l_{j,k-1} + 1) \prod_{i=1}^{k-2}(l_{i,k-2} - l_{j,k-1})}{\prod_{i\neq j=1}^{k-1}(l_{i,k-1} - l_{j,k-1})(l_{i,k-1} - l_{j,k-1} + 1)} \right)^{1/2} |m)_{-j,k-1},$$
$$(2 \leqslant k \leqslant n); \tag{A.4}$$

$$e_{0n}|m) = \sum_{i=1}^{n} \theta_i (-1)^{\theta_1 + \cdots + \theta_{i-1}} (l_{i,n+1} + l_{0,n+1} + 1)^{1/2} \left(\frac{\prod_{k=1}^{n-1}(l_{k,n-1} - l_{i,n+1} - 1)}{\prod_{k\neq i=1}^{n}(l_{k,n+1} - l_{i,n+1})} \right)^{1/2} |m)_{-in}; \tag{A.5}$$

$$e_{n0}|m) = \sum_{i=1}^{n} (1 - \theta_i)(-1)^{\theta_1 + \cdots + \theta_{i-1}} (l_{i,n+1} + l_{0,n+1} + 1)^{1/2} \left(\frac{\prod_{k=1}^{n-1}(l_{k,n-1} - l_{i,n+1} - 1)}{\prod_{k\neq i=1}^{n}(l_{k,n+1} - l_{i,n+1})} \right)^{1/2} |m)_{+in}. \tag{A.6}$$

In all these formulae $l_{ij} = m_{ij} - i$.

It is also useful to know the explicit action of all the odd elements e_{0j} and e_{j0} of $\mathfrak{gl}(1|n)$. This was found in [6, equations (2.25)–(2.26)]:

$$e_{0j}|m) = \sum_{i_n=1}^{n} \sum_{i_{n-1}=1}^{n-1} \cdots \sum_{i_j=1}^{j} \theta_{i_n}(-1)^{\theta_1 + \cdots + \theta_{i_n-1}} \left(l_{i_n,n+1} + l_{0,n+1} + 1\right)^{1/2}$$
$$\times \prod_{r=j+1}^{n} S(i_r, i_{r-1}) \left(\frac{\prod_{k\neq i_{r-1}=1}^{r-1}\left(l_{k,r-1} - l_{i_r,r}\right) \prod_{k\neq i_r=1}^{r}\left(l_{kr} - l_{i_{r-1},r-1} + 1\right)}{\prod_{k\neq i_r=1}^{r}\left(l_{kr} - l_{i_r,r}\right) \prod_{k\neq i_{r-1}=1}^{r-1}\left(l_{k,r-1} - l_{i_{r-1},r-1} + 1\right)} \right)^{1/2}$$
$$\times \left(\prod_{k\neq i_n=1}^{n} \frac{\left(l_{kn} - l_{i_n,n}\right)}{\left(l_{k,n+1} - l_{i_n,n+1}\right)} \right)^{1/2} \left(\frac{\prod_{k=1}^{j-1}\left(l_{k,j-1} - l_{i_j,j}\right)}{\prod_{k\neq i_j=1}^{j}\left(l_{kj} - l_{i_j,j}\right)} \right)^{1/2} |m)_{-i_n,n;-i_{n-1},n-1;\ldots;-i_j,j} \tag{A.7}$$

$$e_{j0}|m) = \sum_{i_n=1}^{n} \sum_{i_{n-1}=1}^{n-1} \cdots \sum_{i_j=1}^{j} \left(1 - \theta_{i_n}\right)(-1)^{\theta_1 + \cdots + \theta_{i_n-1}} \left(l_{i_n,n+1} + l_{0,n+1} + 1\right)^{1/2}$$

$$\times \prod_{r=j+1}^{n} S(i_r, i_{r-1}) \left(\frac{\prod_{k \neq i_{r-1}=1}^{r-1} \left(l_{k,r-1} - l_{i_r,r} - 1\right) \prod_{k \neq i_r=1}^{r} \left(l_{kr} - l_{i_{r-1},r-1}\right)}{\prod_{k \neq i_r=1}^{r} \left(l_{kr} - l_{i_r,r}\right) \prod_{k \neq i_{r-1}=1}^{r-1} \left(l_{k,r-1} - l_{i_{r-1},r-1} - 1\right)} \right)^{1/2}$$

$$\times \left(\prod_{k \neq i_n=1}^{n} \frac{(l_{kn} - l_{i_n,n})}{(l_{k,n+1} - l_{i_n,n+1})} \right)^{1/2} \left(\frac{\prod_{k=1}^{j-1}(l_{k,j-1} - l_{i_j,j} - 1)}{\prod_{k \neq i_j=1}^{j}(l_{kj} - l_{i_j,j})} \right)^{1/2}$$

$$\times |m)_{+i_n,n;+i_{n-1},n-1;\ldots;+i_j,j}, \tag{A.8}$$

where $j = 1, \ldots, n$, each symbol $\pm i_k, k$ attached as a subscript to $|m)$ indicates a replacement $m_{i_k,k} \to m_{i_k,k} \pm 1$, and

$$S(k,l) = \begin{cases} 1 & for \quad k \leqslant l \\ -1 & for \quad k > l. \end{cases} \tag{A.9}$$

References

[1] Lievens S, Stoilova N I and Van der Jeugt J 2006 *J. Math. Phys.* **47** 113504
[2] Cohen-Tannoudji C, Diu B and Laloë F 1977 *Quantum Mechanics* vol 1 (New York: Wiley) complement JV
[3] Kac V G 1977 *Adv. Math.* **26** 8–96
[4] Kac V G 1978 *Lecture Notes Math.* **676** 597–626
[5] Gould M D and Zhang R B 1990 *J. Math. Phys.* **31** 2552–9
[6] King R C, Stoilova N I and Van der Jeugt J 2006 *J. Phys. A: Math. Gen.* **39** 5763–85
[7] Palev T D 1980 *J. Math. Phys.* **21** 1293–8
[8] Audenaert K, Eisert J, Plenio M B and Werner R F 2002 *Phys. Rev.* A **66** 042327
[9] Brun T A and Hartle J B 1999 *Phys. Rev.* D **60** 123503
[10] Eisert J and Plenio M B 2003 *Int. J. Quantum Inf.* **1** 479–506
[11] Halliwell J J 2003 *Phys. Rev.* D **68** 025018
[12] Plenio M B, Hartley J and Eisert J 2004 *New J. Phys.* **6** 36
[13] Kamupingene A H, Palev T D and Tsavena S P 1986 *J. Math. Phys.* **27** 2067–75
[14] Wigner E P 1950 *Phys. Rev.* **77** 711-712
[15] Palev T D 1982 *J. Math. Phys.* **23** 1778–84
     Palev T D 1979 *Czech. J. Phys.* B **29** 91–8
[16] Tolstoy V N, Istomina I F and Smirnov Yu F 1986 *Group Theoretical Methods in Physics* vol 1 (Utrecht: VNU Sci. Press) pp 337–48
[17] Palev T D 1989 *J. Math. Phys.* **30** 1433–42
[18] King R C, Palev T D, Stoilova N I and Van der Jeugt J 2003 *J. Phys. A: Math. Gen.* **36** 4337–62
[19] Wybourne B G 1970 *Symmetry Principles and Atomic Spectroscopy* (New York: Wiley)